Poincaré biextension and ideles on an algebraic curve

Sergey Gorchinskiy*

Abstract

Arbarello, de Concini, and Kac have constructed a central extension of the ideles group on a smooth projective algebraic curve C. We show that this central extension induces the theta-bundle on the class group of degree $g-1$ divisors on C, where g is the genus of the curve C. The other result of the paper is the relation between the product of the norms of the tame symbols over all points of the curve, considered as a pairing on the ideles group, and the Poincaré biextension of the Jacobian of C. As an application we get a new proof of the adelic formula for the Weil pairing.

Introduction

There exists a general ideology which tells that many notions and constructions in algebraic geometry can be translated into the language of certain adelic groups defined for a scheme and some additional data on it, for instance a coherent sheaf (see more details in [4] or [8]). This article provides a new example to this approach.

Let C be a smooth projective curve over a field k, m be an integer prime to $\mathrm{char}(k)$. Consider two divisors D and E on C such that their classes in $\mathrm{Pic}(C)$ belong to the m-torsion. Let $\alpha, \beta \in \mathbf{A}_C^*$ be two ideles such that $\mathrm{div}(\alpha) = D$, $\mathrm{div}(\beta) = E$, and the ideles α^m and β^m are principal, i.e., belong to the subgroup $k(C)^* \subset \mathbf{A}_C^*$; then the Weil pairing $\phi_m([D],[E])$ of the classes of E and D in $\mathrm{Pic}(C)_m$ can be given by the following adelic formula:

$$\phi_m([D],[E]) = (\prod_{x\in C} \mathrm{Nm}_{k(x)/k}[(-1)^{\mathrm{ord}_x(\alpha_x)\mathrm{ord}_x(\beta_x)}(\alpha_x^{\mathrm{ord}_x(\beta_x)}\beta_x^{-\mathrm{ord}_x(\alpha_x)})(x)])^m.$$

The first proof of this formula when C is of any genus appeared in [5]; a more elementary proof was given later by M. Mazo in [6].

On the other hand, Arbarello, de Concini, and Kac have constructed in [1] a certain central extension of the ideles group

$$0 \to k^* \to \widehat{\mathbf{A}}_C^* \xrightarrow{\pi} \mathbf{A}_C^* \to 0.$$

*The author was partially supported by RFFI grants 04-01-00613 and 05-01-00455.

It follows that the commutator in this extension is given by the formula

$$[\pi^{-1}(\alpha),\pi^{-1}(\beta)] = (-1)^{\deg(\alpha)\deg(\beta)}\prod_{x\in C}\mathrm{Nm}_{k(x)/k}[(-1)^{\mathrm{ord}_x(\alpha_x)\mathrm{ord}_x(\beta_x)}(\alpha_x^{\mathrm{ord}_x(\beta_x)}\beta_x^{-\mathrm{ord}_x(\alpha_x)})(x)]$$

for any ideles $\alpha,\beta\in\mathbf{A}_C^*$. The goal of this paper is to find a reason for the apparent similarity of these two formulas. In turns out that there is a close relation between the central extension from [1] and the Poincaré biextension over the Jacobian of C, which defines the Weil pairing. Namely, we prove that the Poincaré biextension is isomorphic to some quotient of the canonically trivial biextension $\Lambda(\widehat{\mathbf{A}}_C^*)$ associated in a usual way with the central extension $\widehat{\mathbf{A}}_C^*$.

The paper is organized as follows. First in section 1 we introduce some notations and recall the construction from [1] of a central extension of the group of ideles on a smooth projective curve C. We show its relation with the theta-bundle on the Picard variety $\mathrm{Pic}^{g-1}(C)$ of degree $g-1$ line bundles on C. Then in section 2 we give some general construction of a quotient biextension associated to a bilinear pairing between abelian groups. In section 3 this construction is applied to the pairing of the ideles group given by the product of the norms of the tame symbols over all points of C. We show that this defines the Poincaré biextension of the Jacobian of C, using results from the previous sections.

The author is grateful to A. N. Parshin for his help and attention to this paper, and also thanks L. Breen, D. Osipov, and M. Mazo for many useful remarks and suggestions.

1 A central extension of ideles and the theta-bundle

Consider a smooth projective curve C of genus g over a field k. Suppose $K=k(C)$ is the field of rational functions on C, $\mathbf{A}_C=\prod'_{x\in C}K_x$ is the ring of adèles on C, and $\mathbf{A}_C^*=\prod'_{x\in C}K_x^*$ is the group of ideles. We put $\mathcal{O}=\prod_{x\in C}\hat{\mathcal{O}}_x\subset\mathbf{A}_C$ and $\mathcal{O}^*=\prod_{x\in C}\hat{\mathcal{O}}_x^*\subset\mathbf{A}_C^*$. There is a natural surjective homomorphism $\mathbf{A}_C^*\to\mathrm{Pic}(C)$, given by the formula $\alpha\mapsto[\mathrm{div}(\alpha)]$, where $\mathrm{div}(\alpha)=\sum_{x\in C}\mathrm{ord}_x(\alpha_x)\cdot x$ and $\mathrm{ord}_x:K_x^*\to\mathbb{Z}$ is the discrete valuation associated with a point $x\in C$. The kernel of this homomorphism is equal to the subgroup $K^*\cdot\mathcal{O}^*\subset\mathbf{A}_C^*$. We put $\deg(\alpha)=\sum_{x\in C}\mathrm{ord}_x(\alpha_x)$. For any two elements $\alpha_x,\beta_x\in K_x^*$, the tame symbol $(\alpha_x,\beta_x)_x\in k(x)^*$ is defined by the formula

$$(\alpha_x,\beta_x)_x=(-1)^{\mathrm{ord}_x(\alpha_x)\mathrm{ord}_x(\beta_x)}(\alpha_x^{\mathrm{ord}_x(\beta_x)}\beta_x^{-\mathrm{ord}_x(\alpha_x)})(x).$$

Let us recall some constructions from [1]. For any two ideles $\alpha,\beta\in\mathbf{A}_C^*$, the subspaces $\alpha\mathcal{O}\subset\mathbf{A}_C$ and $\beta\mathcal{O}\subset\mathbf{A}_C$ are commensurable, i.e., there exists a k-subspace $L\subset\mathbf{A}_C$ such that $L\subset\alpha\mathcal{O}$, $L\subset\beta\mathcal{O}$, and the quotients $(\alpha\mathcal{O})/L$ and $(\beta\mathcal{O})/L$ are finite dimensional. We put $(\alpha\mathcal{O}|\beta\mathcal{O})=\det_k((\alpha\mathcal{O})/L)^{-1}\otimes\det_k((\beta\mathcal{O})/L)$. It is easily seen that this does not depend on the choice of L and $(\alpha\mathcal{O}|\beta\mathcal{O})$ is a well defined one-dimensional space over k. The set $\widehat{\mathbf{A}}_C^*=\{(\alpha,r)|\alpha\in\mathbf{A}_C^*,r\in(\mathcal{O}|\alpha\mathcal{O}),r\neq0\}$ has the structure of a group: the

multiplication in $\widehat{\mathbf{A}}_C^*$ is defined by the canonical isomorphisms $(\mathcal{O}|\beta\mathcal{O}) \xrightarrow{\alpha\cdot} (\alpha\mathcal{O}|\alpha\beta\mathcal{O})$ and $(\mathcal{O}|\alpha\mathcal{O}) \otimes (\alpha\mathcal{O}|\alpha\beta\mathcal{O}) \to (\mathcal{O}|\alpha\beta\mathcal{O})$ for any ideles $\alpha, \beta \in \mathbf{A}_C^*$. Thus we get a central extension

$$1 \to k^* \to \widehat{\mathbf{A}}_C^* \to \mathbf{A}_C^* \to 1. \qquad (*)$$

Recall that for any central extension of an abelian group A by an abelian group N

$$1 \to N \to G \xrightarrow{\pi} A \to 1,$$

the commutator $[g_a, g_b] = g_a g_b g_a^{-1} g_b^{-1} \in N$ depends only in a and b for any two elements $g_a \in \pi^{-1}(a)$, $g_b \in \pi^{-1}(b)$; we put $\langle a, b\rangle = [g_a, g_b]$. The pairing $\langle\cdot,\cdot\rangle$ is skew-symmetric and bilinear. The first assertion is trivial, while for the second one follows from the identity

$$[fg, h] = [f, h][g, h]\mathrm{Ad}(g^{-1}h)(f^{-1})\mathrm{Ad}(hg^{-1})(f)$$

for all elements $f, g, h \in G$, where $\mathrm{Ad}(g)$ is the conjugation by g. Since the commutator of any two elements in G is central, we have $\mathrm{Ad}(g^{-1}h) = \mathrm{Ad}(hg^{-1})$ and $[fg, h] = [f, h][g, h]$.

Remark 1.1. Suppose that the central extension corresponds to the cocycle $\alpha \in H^2(A, N)$ and let $\bar{\alpha}\colon A \times A \to N$ be any representative of this cocycle; then $\langle a, b\rangle = \bar{\alpha}(a, b)\bar{\alpha}(b, a)^{-1}$.

The next result was essentially proved in [1].

Theorem 1.1. *For all $\alpha, \beta \in \mathbf{A}_C^*$, the commutator of their liftings to the group $\widehat{\mathbf{A}}_C^*$ is equal up to sign to the product of the norms of the tame symbols:*

$$\langle\alpha, \beta\rangle = (-1)^{\deg(\alpha)\deg(\beta)} \prod_{x \in C} \mathrm{Nm}_{k(x)/k}[(\alpha_x, \beta_x)_x].$$

There is a cohomological interpretation of the one-dimensional space $(\mathcal{O}|\alpha\mathcal{O})$, $\alpha \in \mathbf{A}_C^*$. Namely, for any k-subspace $L \subset \mathbf{A}_C$, consider the *adelic complex*

$$\mathbf{A}(L)^\bullet\colon 0 \to K \oplus L \to \mathbf{A}_C \to 0,$$

where the differential is given by the formula $(f, \{f_x\}) \mapsto \{f - f_x\}$ for $f \in K$, $\{f_x\} \in L$. Let $\mathbf{D}(L) = \det_k H^0(\mathbf{A}(L)^\bullet) \otimes \det_k H^1(\mathbf{A}(L)^\bullet)^{-1}$ be the determinant of cohomology of this complex. We claim that there is a canonical isomorphism

$$\mathbf{D}(\mathcal{O}) \otimes (\mathcal{O}|\alpha\mathcal{O}) \cong \mathbf{D}(\alpha\mathcal{O}).$$

Indeed, let $L \subset \mathbf{A}_C$ be a k-subspace such that $L \subset \mathcal{O}$, $L \subset \alpha\mathcal{O}$, and the k-spaces $\mathcal{O}/L$ and $(\alpha\mathcal{O})/L$ are finite-dimensional; then the natural embeddings of complexes $\mathbf{A}(L)^\bullet \hookrightarrow \mathbf{A}(\mathcal{O})^\bullet$ and $\mathbf{A}(L)^\bullet \hookrightarrow \mathbf{A}(\alpha\mathcal{O})^\bullet$ imply the needed result. In other words, $(\mathcal{O}|\alpha\mathcal{O}) = \mathrm{Hom}_k(\mathbf{D}(\mathcal{O}), \mathbf{D}(\alpha\mathcal{O}))$.

This interpretation allows to construct a canonical element $\widehat{f} \in (\mathcal{O}|f\mathcal{O})\backslash\{0\} \subset \widehat{\mathbf{A}}_C^*$ for any $f \in K^*$, using the isomorphism of complexes $\mathbf{A}(\mathcal{O})^\bullet \xrightarrow{\cdot f} \mathbf{A}(f\mathcal{O})^\bullet$, which leads to the isomorphism of one-dimensional k-spaces $\mathbf{D}(\mathcal{O}) \to \mathbf{D}(f\mathcal{O})$. It is easy to check that the assignment $f \mapsto \widehat{f}$ gives a splitting of the central extension $(*)$ over the subgroup $K^* \subset \mathbf{A}_C^*$, i.e, we have $\widehat{f} \cdot \widehat{g} = \widehat{f \cdot g}$ for all $f, g \in K^*$.

Remark 1.2. As shown in [1], combining the splitting of $(*)$ over K^* with the formula from Theorem 1.1, we get the Weil reciprocity law on C.

Further, for any element $u \in \mathcal{O}^* \subset \mathbf{A}_C^*$, we have $\mathcal{O} = u\mathcal{O}$, hence the space $(\mathcal{O}|u\mathcal{O})$ is canonically isomorphic to k. This defines the splitting $\mathcal{O}^* \to \widehat{\mathbf{A}}_C^*$, $u \mapsto \widetilde{u}$ of the extension $(*)$. We denote the splittings over K^* and $\mathcal{O}^*$ in the different ways because they do not coincide on the intersection $k^* = K^* \cap \mathcal{O}^* \subset \mathbf{A}_C^*$. Indeed, for a constant $c \in k^*$, we have $\widehat{c} = c^{\chi(\mathbf{A}(\mathcal{O})^\bullet)} \cdot \widetilde{c}$. To compute the Euler characteristic $\chi(\mathbf{A}(\mathcal{O})^\bullet)$, we give the following geometrical interpretation of the complexes $\mathbf{A}(\alpha\mathcal{O})^\bullet$, $\alpha \in \mathbf{A}_C^*$.

For any invertible sheaf $\mathcal{L}$ on C, consider the *adelic complex*

$$\mathbf{A}(C,\mathcal{L})^\bullet\colon 0 \to \mathcal{L}_\eta \oplus \prod_{x \in C} \hat{\mathcal{L}}_x \to \prod_{x\in C}{}' \hat{\mathcal{L}}_x \otimes_{\hat{\mathcal{O}}_x} K_x \to 0,$$

where η is the generic point of C, $\hat{\mathcal{L}}_x = \mathcal{L}_x \otimes_{\mathcal{O}_x} \hat{\mathcal{O}}_x$ and $\prod'$ is the adelic product (for more details see [4] or [8]). It is known that there are canonical isomorphisms $H^i(C,\mathcal{L}) \cong H^i(\mathbf{A}(C,\mathcal{L})^\bullet)$ for $i = 0, 1$. On the other hand, there is an equality of complexes

$$\mathbf{A}(C, \mathcal{O}_C(D))^\bullet = \mathbf{A}(\alpha\mathcal{O})^\bullet$$

for any idele $\alpha \in \mathbf{A}_C^*$, where $D = -\mathrm{div}(\alpha)$ and for any open subset $U \subset C$, the group $\mathcal{O}_C(D)(U)$ consists of all functions $f \in K^*$ such that $(\mathrm{div}(f) + D)|_U \geq 0$. Thus there are canonical isomorphisms $H^i(\mathbf{A}(\alpha\mathcal{O})^\bullet) \cong H^i(C, \mathcal{O}_C(D))$, $\mathbf{D}(\alpha\mathcal{O}) \cong \det R\Gamma(C, \mathcal{O}_C(D))$, and $(\mathcal{O}|\alpha\mathcal{O}) \cong \det R\Gamma(C,\mathcal{O}_C(D)) \otimes \det R\Gamma(C, \mathcal{O}_C)^{-1}$, where $i = 0, 1$, $\alpha \in \mathbf{A}_C^*$, and $D = -\mathrm{div}(\alpha)$. In particular, we see that $\chi(\mathbf{A}(\mathcal{O})^\bullet) = 1 - g$, where g is the genus of the curve C and therefore the splittings of $(*)$ over K^* and $\mathcal{O}^*$ do not coincide in general on the intersection $k^* = K^* \cap \mathcal{O}^*$.

Lemma 1.1.

(i) For any elements $f \in K^*$ and $\alpha \in \mathbf{A}_C^*$, the following diagram commutes

$$\begin{array}{ccc}
(\mathcal{O}|\alpha\mathcal{O}) & \rightarrow & \det R\Gamma(C,\mathcal{O}_C(D)) \otimes \det R\Gamma(C,\mathcal{O}_C)^{-1} \\
\downarrow \widehat{f}\cdot & & \downarrow \\
(\mathcal{O}|f\alpha\mathcal{O}) & \rightarrow & \det R\Gamma(C,\mathcal{O}_C(D - \mathrm{div}(f))) \otimes \det R\Gamma(C,\mathcal{O}_C)^{-1}
\end{array}$$

where $D = -\mathrm{div}(\alpha)$, the horizontal arrows are canonical isomorphisms, the first vertical arrow is multiplication on the left by $\widehat{f}$ in the group $\widehat{\mathbf{A}}_C^*$, and the second vertical arrow is defined by the canonical isomorphism of invertible sheaves $\mathcal{O}_C(D) \cong \mathcal{O}_C(D - (f))$ which is multiplication by f in K^*.

(ii) For any elements $u \in \mathcal{O}^*$ and $\alpha \in \mathbf{A}_C^*$, the following diagram commutes

$$\begin{array}{ccc}
(\mathcal{O}|\alpha\mathcal{O}) & \rightarrow & \det R\Gamma(C,\mathcal{O}_C(D)) \otimes \det R\Gamma(C,\mathcal{O}_C)^{-1} \\
\downarrow \cdot\widetilde{u} & & \downarrow id \\
(\mathcal{O}|\alpha u\mathcal{O}) & \rightarrow & \det R\Gamma(C,\mathcal{O}_C(D))) \otimes \det R\Gamma(C,\mathcal{O}_C)^{-1}
\end{array}$$

where $D = -\mathrm{div}(\alpha)$, the horizontal arrows are canonical isomorphisms, the first vertical arrow is multiplication on the right by $\widetilde{u}$ in the group $\widehat{\mathbf{A}}_C^*$, and the second vertical arrow is the identity.

Proof. Consider an arbitrary element $r \in (\mathcal{O}|\alpha\mathcal{O})\backslash\{0\}$ and the commutative diagram:

$$\begin{array}{ccc} \mathbf{D}(\mathcal{O}) & \xrightarrow{r} & \mathbf{D}(\alpha\mathcal{O}) \\ \downarrow f\cdot & & \downarrow f\cdot \\ \mathbf{D}(f\mathcal{O}) & \xrightarrow{f(r)} & \mathbf{D}(f\alpha\mathcal{O}). \end{array}$$

By definition, the composition of the lower triangle in this diagram, i.e., the diagonal, is equal to $\widehat{f} \cdot r \in (\mathcal{O}|f\alpha\mathcal{O})$. On the other hand, the composition of the upper triangle corresponds to the identification of $(\mathcal{O}|\alpha\mathcal{O})$ with $(\mathcal{O}|f\alpha\mathcal{O})$ via the isomorphism $\det R\Gamma(C, \mathcal{O}_C(D)) \xrightarrow{f\cdot} \det R\Gamma(C, \mathcal{O}_C(D - \mathrm{div}(f)))$ and this proves (i). The proof of (ii) is analogous. $\square$

For any integer $n \in \mathbb{Z}$, let $(\mathbf{A}_C^*)^n$ be the set of ideles α such that $\deg(-\mathrm{div}(\alpha)) = n$ and let $(\widehat{\mathbf{A}}_C^*)^n$ be the preimage of $(\mathbf{A}_C^*)^n$ in $\widehat{\mathbf{A}}_C^*$. Let Θ be the line bundle on $\mathrm{Pic}^{g-1}(C)$ whose fiber over an isomorphism class L of degree $g-1$ line bundles on C is given by $\Theta|_L = \det R\Gamma(C, \mathcal{L}) \otimes \det R\Gamma(C, \mathcal{O})^{-1}$, where $\mathcal{L}$ is any representative in L. Note that since $\chi(C, \mathcal{L}) = 0$, this one-dimensional k-space is well defined.

Remark 1.3. It is known that the line bundle Θ is isomorphic to the line bundle associated with the theta-divisor on $\mathrm{Pic}^{g-1}(C)$ (see [7]).

The next result is a direct consequence of Lemma 1.1.

Proposition 1.1. *There is a well defined action of the group $K^* \cdot \mathcal{O}^*$ on the set $(\widehat{\mathbf{A}}_C^*)^{g-1}$ given by the formula $(fu)(h) = \widehat{f} \cdot h \cdot \widetilde{u}$ for all $f \in K^*$, $u \in \mathcal{O}^*$, and $h \in (\widehat{\mathbf{A}}_C^*)^{g-1}$; this action commutes with the natural action of $K^* \cdot \mathcal{O}^*$ on $(\mathbf{A}_C^*)^{g-1}$. Moreover, there is a canonical isomorphism of k^*-torsors on* $\mathrm{Pic}^{g-1}(C)$

$$K^*\backslash(\widehat{\mathbf{A}}_C^*)^{g-1}/\mathcal{O}^* \cong \Theta\backslash\{0\},$$

where we identify $K^*\backslash(\mathbf{A}_C^*)^{g-1}/\mathcal{O}^*$ with $\mathrm{Pic}^{g-1}(C)$ *via the map $\alpha \mapsto -\mathrm{div}(\alpha)$, $\alpha \in \mathbf{A}_C^*$.*

2 Construction of a quotient biextension

For all groups below, we write the group law in the multiplicative way. See more details on biextensions in [9] and [2]. Let A, A', N be abelian groups, $B, B' \subset A$, $C, C' \subset A$ be subgroups, and let $\langle\cdot,\cdot\rangle : A \times A \to N$ be a bilinear pairing such that $\langle B, B'\rangle = 1$, $\langle C, C'\rangle = 1$, $\langle B \cap C, A'\rangle = 1$, and $\langle A, B' \cap C'\rangle = 1$. Let T be the trivial biextension of (A, A') by N. By $T|_{(a,a')}$ denote the fiber of T over $(a, a') \in A \times A'$. For all elements $a \in A$, $bc \in B \cdot C$, $a' \in A'$, and $b'c' \in B' \cdot C'$, consider the isomorphism $T|_{(a,a')} \to T|_{(abc,a'b'c')}$ that is equal to multiplication by $\langle a', c\rangle\langle b', a\rangle\langle b', c\rangle \in N$. It is readily seen that the last expression does not depend on the decompositions bc and $b'c'$ and it can be checked that this defines an action of the group $(B \cdot C) \times (B' \cdot C')$ on T, which commutes with the natural action on $A \times A'$. Moreover, this action commutes with the biextension structure on T and we get the quotient biextension $P = T/((B \cdot C) \times (B' \cdot C'))$ of $(A/(B \cdot C), A'/(B' \cdot C'))$ by N.

Remark 2.1. Given a central extension $1 \to N \to G \to A \to 1$, one has a canonically trivial biextension $\Lambda(G) = m^*G \wedge p_1^*G^{-1} \wedge p_2^*G^{-1}$ of (A, A) by N, where p_1, p_2, m, and $\wedge$ denote, respectively, projection on the first multiple, projection on the second multiple, multiplication in the group A, and product in the category of N-torsors on $A \times A$. If the extension splits over subgroups $B, C \subset A$, then the commutator pairing $\langle \cdot, \cdot \rangle$ satisfies the above condition with $A' = A$, $B' = B$, and $C = C'$ if we suppose also that $\langle B \cap C, A \rangle = 1$. Let the assignments $b \mapsto \widehat{b}$ and $c \mapsto \widetilde{c}$ be splittings of the given central extension over B and C, respectively; then there is an action of $B \times C$ on G given by the formula $g \mapsto \hat{b} g \widetilde{c}$. This naturally induces the action of $(B \times C) \times (B \times C)$ on $\Lambda(G)$. An explicit calculation shows that this action factors through $(B \cdot C) \times (B \cdot C)$ and coincides with the one defined above.

Let D and D' be two abelian groups. Recall that for any biextension P of (D, D') by N and for any integer $m \in \mathbb{Z}$, $m \geq 1$, one defines the *Weil pairing* $\phi_m : A_m \times A_m \to N_m$ in the following way: for any $(d, d') \in D_m \times D'_m$, the element $\phi_m(d, d')$ is equal to the composition given by the diagram of N-torsors

$$\begin{array}{ccc} P^{\wedge m}|_{(d,d')} & \longrightarrow & P|_{(d^m,d')} \\ \uparrow & & \downarrow \\ P|_{(d,(d')^m)} & \longleftarrow & P|_{(1,1)}, \end{array}$$

where the arrows are natural isomorphisms of N-torsor defined by the biextension structure on P.

Proposition 2.1. *Let $A, A', B, B', C, C', \langle \cdot, \cdot \rangle, T, P$ be as in the beginning of this section and let $a \in A, a' \in A', b \in B, b' \in B, c \in C, c' \in C'$ be such that $a^m = bc$ and $(a')^m = b'c'$; then we have*

$$\phi_m(\bar{a}, \bar{a}') = \langle b', a \rangle \langle a', c \rangle^{-1},$$

where $\bar{a} \in A/(B \cdot C)$ and $\bar{a}' \in A'/(B' \cdot C')$ are the classes corresponding to a and a', respectively.

Proof. By construction, the pull-back of the biextension P from $A/(B \cdot C) \times A'/(B' \cdot C')$ to $A \times A'$ is isomorphic to the trivial biextension T. Therefore the pull-back of the diagram defining the Weil pairing $\phi_m(\bar{a}, \bar{a}')$ is the diagram

$$\begin{array}{ccc} T^{\wedge m}|_{(a,a')} & \xrightarrow{id} & T|_{(bc,a')} \\ \uparrow id & & \downarrow \langle a', c \rangle^{-1} \\ T|_{(a,b'c')} & \xleftarrow{\langle b',a \rangle} & T|_{(1,1)}. \end{array}$$

This concludes the proof. ∎

3 Tame symbols and the Poincaré biextension

As before, let C be a smooth projective curve of genus g over a field k. Let us recall a construction of the Poincaré biextension $\mathcal{P}$ of $(\mathrm{Pic}^0(C), \mathrm{Pic}^0(C))$ by k^* (see [3] and [7]).

For all isomorphism classes L, M of degree zero line bundles on C, we put $\mathcal{P}|_{(L,M)} = (\mathcal{L}, \mathcal{M})$, where

$$(\mathcal{L}, \mathcal{M}) = (\det R\Gamma(C, \mathcal{L} \otimes \mathcal{M}) \otimes \det R\Gamma(C, \mathcal{L})^{-1} \otimes \det R\Gamma(C, \mathcal{M})^{-1} \otimes \det R\Gamma(C, \mathcal{O}_C)) \backslash \{0\}$$

and $\mathcal{L}$ and $\mathcal{M}$ are any representatives from L and M, respectively. Since $\chi(C, \mathcal{L} \otimes \mathcal{M}) = \chi(C, \mathcal{L}) = \chi(C, \mathcal{M}) = 1 - g$, this one-dimensional k-space is well defined and $\mathcal{P}$ is a k^*-torsor on $\mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$. To define a biextension structure on $\mathcal{P}$, consider the pull-back $p^*\mathcal{P}$ of $\mathcal{P}$ with respect to the natural map $p : \mathrm{Pic}^0(C) \times \mathrm{Div}^0(C) \to \mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$ given by the formula $(L, D) \mapsto (L, [\mathcal{O}_C(D)])$. There is a canonical isomorphism $\varphi : p^*\mathcal{P} \cong \mathcal{P}'$ of k^*-torsors on $\mathrm{Pic}^0(C) \times \mathrm{Div}^0(C)$, where $\mathcal{P}'$ is the biextension of $(\mathrm{Pic}^0(C), \mathrm{Div}^0(C))$ by k^* defined by the formula $\mathcal{P}'|_{(L,D)} = (\bigotimes\limits_{x \in C} \mathcal{L}|_x^{\otimes \mathrm{ord}_x(D)}) \backslash \{0\}$, where $\mathcal{L}$ is any representative from the class L. Thus φ induces a biextension structure on $p^*\mathcal{P}$ and it turns out that it descends to a biextension structure on $\mathcal{P}$.

Now we put $A = A' = (\mathbf{A}_C^*)^0$, $B = B' = K^*$, $C = C' = \mathcal{O}^*$, $N = k^*$, and $\langle \alpha, \beta \rangle = \prod\limits_{x \in C} \mathrm{Nm}_{k(x)/k}[(\alpha_x, \beta_x)_x]$. It is readily seen that the conditions from the beginning of section 2 are satisfied, hence we get a biextension P of $(\mathrm{Pic}^0(C), \mathrm{Pic}^0(C))$ by k^*.

Theorem 3.1. *The biextension P is canonically isomorphic to the Poincaré biextension $\mathcal{P}$ of $(\mathrm{Pic}^0(C), \mathrm{Pic}^0(C))$ by k^*.*

Proof. Let $\pi' : (\mathbf{A}_C^*)^0 \times (\mathbf{A}_C^*)^0 \to \mathrm{Pic}^0(C) \times \mathrm{Div}^0(C)$ be the homomorphism given by the formula $(\alpha, \beta) \mapsto ([\mathcal{O}_C(-\mathrm{div}(\alpha))], -\mathrm{div}(\beta))$ and let $\pi = p \circ \pi' : (\mathbf{A}_C^*)^0 \times (\mathbf{A}_C^*)^0 \to \mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$.

It follows from section 1 that there is a canonical isomorphism $\psi : \Lambda((\widehat{\mathbf{A}}_C^*)^0) \to \pi^*\mathcal{P}$ of k^*-torsors over $(\mathbf{A}_C^*)^0 \times (\mathbf{A}_C^*)^0$. Combining Theorem 1.1, Lemma 1.1, and Remark 2.1, we see that the natural action of $(K^* \cdot \mathcal{O}^*) \times (K^* \cdot \mathcal{O}^*)$ on $\pi^*\mathcal{P}$ commutes via ψ with the action of $(K^* \cdot \mathcal{O}^*) \times (K^* \cdot \mathcal{O}^*)$ on the canonically trivial biextension $\Lambda((\widehat{\mathbf{A}}_C^*)^0)$ described in the beginning of section 2 and defining the biextension P. Therefore P is isomorphic to $\mathcal{P}$ as a k^*-torsor on $\mathrm{Pic}^0(C) \times \mathrm{Pic}^0(C)$ and it remains to check that the canonical isomorphism ψ commutes with the biextension structures.

Note that the pull-back $(\pi')^*\mathcal{P}'$ has a canonical trivialization given by the assignment

$$(\alpha, \beta) \mapsto \bigotimes_{x \in C} \bar{\alpha}_x^{\otimes(-\mathrm{ord}_x(\beta_x))} \in \bigotimes_{x \in C} (\alpha_x \mathcal{O}_x / \mathfrak{m}_x \alpha_x \mathcal{O}_x)^{\otimes(-\mathrm{ord}_x(\beta_x))} = (\pi')^*\mathcal{P}'|_{(\alpha,\beta)}.$$

Thus it suffices to check that the composition $(\pi')^*\varphi \circ \psi : \Lambda((\widehat{\mathbf{A}}_C^*)^0) \to (\pi')^*\mathcal{P}'$ sends one trivialization to the other.

Let us recall the explicit form of the isomorphism φ. Take a pair $([\mathcal{L}], D) \in \mathrm{Pic}^0(C) \times \mathrm{Div}^0(C)$. First, suppose that $D \geq 0$; then the exact sequences of sheaves

$$0 \to \mathcal{L} \to \mathcal{L}(D) \to \mathcal{L}(D)|_D \to 0,$$

$$0 \to \mathcal{O} \to \mathcal{O}(D) \to \mathcal{O}(D)|_D \to 0$$

lead to the isomorphism $\mu : (\mathcal{L}, \mathcal{O}_C(D)) \to \det_k(\mathcal{L}(D)|_D) \otimes \det_k(\mathcal{O}(D)|_D)^{-1}$. Further, by induction on the degree of D, one establishes a canonical isomorphism

$$\nu \colon \mathrm{Hom}_k(\det_k(\mathcal{O}_C(D)|_D), \det_k(\mathcal{L}(D)|_D)) \cong \bigotimes_{x \in C} \mathcal{L}|_x^{\otimes \mathrm{ord}_x(D)}.$$

The isomorphism φ equals the composition $\nu \circ \mu$.

Suppose that $\{s_x\} \in \prod_{x \in C} \mathcal{L}_x$ is a collection of local sections such that $\bar{s}_x \neq 0$ for all $x \in C$, where $\bar{s}_x \in \mathcal{L}|_x$ is the value at a point x of a section $s_x \in \mathcal{L}_x$. Then the determinant of the isomorphism $\bigotimes_{x \in |D|} s_x \colon \mathcal{O}_C(D)|_D \to \mathcal{L}(D)|_D$ is mapped under ν to the product $\bigotimes_{x \in C} \bar{s}_x^{\otimes \mathrm{ord}_x(D)}$, where $|D|$ is the support of the divisor D.

Now consider the pull-backs of μ and ν with respect to π'. Let $(\alpha, \beta) \in (\mathbf{A}_C^*)^0 \times (\mathbf{A}_C^*)^0$ be such that $\pi'(\alpha, \beta) = ([\mathcal{L}], D)$. We may assume that $\mathcal{L} = \mathcal{O}_C(-\mathrm{div}(\alpha))$. Then the map $(\pi')^* \mu \circ \psi$ is the natural isomorphism

$$\Lambda((\widehat{\mathbf{A}}_C^*)^0)|_{(\alpha,\beta)} = (\mathcal{O}|\alpha\beta\mathcal{O}) \otimes (\mathcal{O}|\alpha\mathcal{O})^{-1} \otimes (\mathcal{O}|\beta\mathcal{O})^{-1} \to (\alpha\mathcal{O}|\alpha\beta\mathcal{O}) \otimes (\mathcal{O}|\beta\mathcal{O})^{-1}$$

that follows from the exact sequences of complexes

$$0 \to \mathbf{A}(\alpha\mathcal{O}) \to \mathbf{A}(\alpha\beta\mathcal{O}) \to (\alpha\mathcal{O}|\alpha\beta\mathcal{O}) \to 0,$$

$$0 \to \mathbf{A}(\mathcal{O}) \to \mathbf{A}(\beta\mathcal{O}) \to (\mathcal{O}|\beta\mathcal{O}) \to 0.$$

Therefore the isomorphism $(\pi')^* \mu \circ \psi$ takes the canonical element in $\Lambda((\widehat{\mathbf{A}}_C^*)^0)|_{(\alpha,\beta)}$ to the element $\det(\alpha) \in \mathrm{Hom}_k((\mathcal{O}|\beta\mathcal{O}), (\alpha\mathcal{O}|\alpha\beta\mathcal{O}))$ that equals to the determinant of the isomorphism $(\mathcal{O}|\beta\mathcal{O}) \xrightarrow{\alpha\cdot} (\alpha\mathcal{O}|\alpha\beta\mathcal{O})$. Further, the idele α defines a collection of local sections $\{\alpha_x\} \in \prod_{x \in C} \mathcal{L}_x$, hence $(\pi')^* \nu(\det(\alpha)) = \bigotimes_{x \in C} \bar{\alpha}_x^{\otimes(-\mathrm{ord}_x(\beta_x))}$. Thus we have treated the case when the divisor D is effective.

One considers the case when $E = -D \geq 0$ in the same way, using the exact sequences of sheaves

$$0 \to \mathcal{L}(-E) \to \mathcal{L} \to \mathcal{L}|_E \to 0,$$

$$0 \to \mathcal{O}_C(-E) \to \mathcal{O}_C \to \mathcal{O}_C|_E \to 0.$$

The case of an arbitrary divisor $D - E$, where $D, E \geq 0$, can be reduced to these two cases, using the embeddings of sheaves $\mathcal{L}(-E) \subset \mathcal{L}$ and $\mathcal{L}(-E) \subset \mathcal{L}(D-E)$ (respectively, $\mathcal{O}_C(-E) \subset \emptyset$ and $\mathcal{O}_C(-E) \subset \mathcal{O}_C(D-E)$), whose pull-back with respect to π' will correspond to the choice of a common k-subspace in the commensurable spaces $\alpha\mathcal{O}$ and $\alpha\beta\mathcal{O}$ (respectively, $\mathcal{O}$ and $\beta\mathcal{O}$) when defining the one-dimensional space $(\alpha\mathcal{O}|\alpha\beta\mathcal{O})$ (respectively, $(\mathcal{O}|\beta\mathcal{O})$). $\square$

Remark 3.1. One can also descend a symmetric structure from the trivial biextension T of $((\mathbf{A}_C^*)^0, (\mathbf{A}_C^*)^0)$ to the biextension P and check this coincides with the natural symmetric structure on $\mathcal{P}$.

Recall that for a natural number m prime to $\mathrm{char}(k)$, the Weil pairing $\phi_m : \mathrm{Pic}^0(C)_m \times \mathrm{Pic}^0(C)_m \to \mu_m$ is the Weil pairing in the above sense associated with the Poincaré biextension $\mathcal{P}$ (see [7]). Combining Proposition 2.1 with Theorem 3.1, we get the following adelic formula for the Weil pairing.

Corollary 3.1. *Let $\alpha, \alpha' \in \mathbf{A}_C^*$ be two ideles such that $\alpha^m = fu$ and $(\alpha')^m = f'u'$, where $f, f' \in K^*$, $u, u' \in \mathcal{O}^*$, and let $\mathcal{L} = \mathcal{O}_C(-\mathrm{div}(\alpha))$, $\mathcal{M} = \mathcal{O}_C(-\mathrm{div}(\alpha'))$; then we have*

$$\phi_m(\mathcal{L}, \mathcal{M}) = \prod_{x \in C} \mathrm{Nm}_{k(x)/k}[(f, \alpha'_x)_x (\alpha_x, u'_x)_x^{-1}].$$

Remark 3.2. If the divisors $D = -\mathrm{div}(\alpha)$ and $D' = -\mathrm{div}(\alpha')$ do not intersect, then we have $\phi_m(\mathcal{L}, \mathcal{M}) = f'(D) \cdot f^{-1}(D')$. The equivalence of this definition of the Weil pairing with usual one was first shown by Howe in [5].

Remark 3.3. Suppose that the ground field k is algebraically closed; then the group $\hat{\mathcal{O}}_x$ is m-divisible for any closed point $x \in C$ and any integer m prime to $\mathrm{char}(k)$. Therefore, given the divisors D and D', one may choose the ideles α and α' such that $\alpha^m = f$ and $(\alpha')^m = f'$, where $f, f' \in K^*$. Then $\phi_m(\mathcal{L}, \mathcal{M}) = \prod_{x \in C} \mathrm{Nm}_{k(x)/k}[(\alpha_x, \alpha'_x)_x^m]$. The coincidence of this formula with the definition of the Weil pairing via biextensions was also directly explained by Mazo in [6].

Bibliography

[1] E. Arbarello, C. de Concini, V. G. Kac, "The Infinite Wedge Representation and the Reciprocity Law for Algebraic Curves", *Proc. Sympos. Pure Math.* **49**:1 (1989), 171–190.

[2] L. Breen, "Fonctions thêta et théorème du cube", *Lecture Notes in Mathematics*, **980** (1983).

[3] P. Deligne, "Le determinant de la cohomologie", *Contemp. Math.*, **67** (1987), 93–177.

[4] T. Fimmel, A. N. Parshin, "An introduction to the higher adelic theory", preprint (1999).

[5] E. W. Howe, "The Weil pairing and the Hilbert symbol", *Math. Ann.*, **305**:2 (1996), 387–392.

[6] M. Mazo, "Weil pairing and tame symbols", *Math. Notes*, **77**:5 (2005).

[7] L. Moret-Bailly, "Métriques Permises", *Séminaire sur les Pinceaux Arithmétiques: La Conjecture de Mordell (Paris, 1983/84)*, Astérisque **127**, Soc. Math. France, Paris, 1985, 29–87.

[8] A. N. Parshin, “Chern classes, adeles and L-functions”, *J. Reine Angew. Math.*, **341** (1983), 174–192.

[9] “Groupes de Monodromie en Géométrie Algébrique” (SGA 7), *Lecture Note in Mathmatics*, *288* (1972).